EXPONENTIAL GROWTH OF HOMOLOGICAL TORSION FOR TOWERS OF CONGRUENCE SUBGROUPS OF BIANCHI GROUPS

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Abstract. In this paper we prove that for suitable sequences of congruence subgroups of Bianchi groups, including the standard exhaustive sequences of a congruence subgroup, and even symmetric powers of the standard representation of $\mathrm{SL}_2(\mathbb{C})$ the size of the torsion part in the first homology grows exponentially. This extends results of Bergeron and Venkatesh to a case of non-uniform lattices.

1. Introduction

The aim of the present paper is to extend some recent results obtained by Bergeron and Venkatesh for cocompact arithmetic groups to the case of Bianchi groups which are not cocompact. Thus we shall firstly recall some of the results of Bergeron and Venkatesh. Let $\mathbf{G}$ be a semisimple connected linear algebraic group defined over $\mathbb{Q}$ and let Γ be a congruence subgroup of $\mathbf{G}(\mathbb{Q})$. Assume that Γ is cocompact or equivalently that $\mathbf{G}$ is anisotropic over $\mathbb{Q}$. Let $G := \mathbf{G}(\mathbb{R})$ be the real points of $\mathbf{G}$ and let K be a maximal compact subgroup of G. Then one can form the globally symmetric space $\widetilde{X} := G/K$ and the locally symmetric space $X := \Gamma\backslash\widetilde{X}$. Let $\delta(\widetilde{X}) := \mathrm{rank}_{\mathbb{C}}(G) - \mathrm{rank}_{\mathbb{C}}(K)$

Now let L be an arithmetic Γ-module. This means that there exists a $\mathbb{Q}$-rational representation ρ of $\mathbf{G}$ on a finite-dimensional $\mathbb{Q}$-vector space $V_{\mathbb{Q}}$ such that L is a $\mathbb{Z}$-lattice in $V_{\mathbb{Q}}$ which is invariant under Γ. Restricting ρ to Γ, one can form the homology groups $H_q(\Gamma, L)$. These are finitely generated abelian groups and one lets $H_q(\Gamma, L)_{tors}$ be their torsion part. If $\delta(\widetilde{X}) = 1$ and $\widetilde{X}$ is odd-dimensional, for arithmetic reasons one expects these groups to be large, see [BV] and also the recent book of Calegari and Venkatesh [CV]. Bergeron and Venkatesh verified this conjecture in the following way: If the representation $\rho_{\mathbb{R}}$ of G on $V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$ associated to ρ is strongly acyclic, which means that it admits a uniform spectral gap near zero (see [BV, page 15]), then for every sequence Γ_i of finite index (congruence) subgroups of Γ for which the injectivity radius goes to infinity, they proved that

$$\liminf_{i\to\infty} \sum_{q\equiv d(2)} \frac{\log|H_q(\Gamma_i, L)_{tors}|}{[\Gamma : \Gamma_i]} \geq c_{\widetilde{X}}(\rho)\,\mathrm{vol}(X), \tag{1.1}$$

see [BV, Theorem 1.4], where $d = \frac{\dim\widetilde{X}-1}{2}$. Here $c_{\widetilde{X}}(\rho)$ is a strictly positive constant which depends only on the globally symmetric space $\widetilde{X}$. More precisely, if X is any (compact) manifold of the form $X = \Gamma\backslash\widetilde{X}$, where Γ is a discrete, torsion-free subgroup of G, and if one denotes by $\log T_X^{(2)}(\rho)$ the L^2-torsion of X with respect to the flat bundle defined by

the restricion of $\rho_{\mathbb{R}}$ to Γ, then one has

$$\log T_X^{(2)}(\rho) = (-1)^d c_{\widetilde{X}}(\rho)\operatorname{vol}(X). \tag{1.2}$$

Furthermore, Bergeron and Venkatesh conjectured that in the asymptotic formula (1.1) only the d-th part contributes to the growth of the torsion in the limit, i.e. they conjectured that for every q the limit

$$\lim_{i\to\infty} \frac{\log |H_q(\Gamma_i, L)_{tors}|}{[\Gamma : \Gamma_i]}$$

exists, that for $q \neq d$ it is zero and that for $q = d$ it equals $c_{\widetilde{X}}(\rho)\operatorname{vol}(X)$, see [BV, Conjecture 1.3]. This conjecture has been verified by Bergeron and Venkatesh in [BV] for cocompact arithmetic subgroups of $\mathrm{SL}_2(\mathbb{C})$ and for strongly acyclic representations.

In the present paper we prove a modified analog of the result (1.1) of Bergeron and Venkatesh for congruence subgroups of Bianchi groups, which are non-uniform lattices of $\mathrm{SL}_2(\mathbb{C})$, and for certain strongly acyclic representations ρ. The group $\mathrm{SU}(2)$ is a maximal compact subgroup of $\mathrm{SL}_2(\mathbb{C})$ and the globally symmetric space $\widetilde{X} = \mathrm{SL}_2(\mathbb{C})/\mathrm{SU}(2)$ satisfies $\delta(\widetilde{X}) = 1$. Moreover, $\widetilde{X}$ is isometric to the 3-dimensional hyperbolic space $\mathbb{H}^3$. Let $F = \mathbb{Q}(\sqrt{-D})$, $D \in \mathbb{N}$ square-free, be an imaginary quadratic number field and let $\mathcal{O}_D$ be its ring of integers. Let $\Gamma(D) := \mathrm{SL}_2(\mathcal{O}_D)$. This is a discrete subgroup of $\mathrm{SL}_2(\mathbb{C})$. It is not cocompact, but the covolume $\operatorname{vol}(\Gamma(D)\backslash\mathbb{H}^3)$ is finite. If $\mathfrak{a}$ is a non-zero ideal in $\mathcal{O}_D$, we let $\Gamma(\mathfrak{a})$ be the associated principal congruence subgroup of level $\mathfrak{a}$. This is a finite index subgroup of $\Gamma(D)$ which is torsion-free for $N(\mathfrak{a})$ sufficiently large. A subgroup Γ_0 of $\Gamma(D)$ is called a congruence subgroup if it contains $\Gamma(\mathfrak{a})$ for some non-zero ideal $\mathfrak{a}$.

For $m \in \mathbb{N}$ we let $\rho(m)$ denote the $2m$-th symmetric power of the standard representation of $\mathrm{SL}_2(\mathbb{C})$ on $V(m) := \mathrm{Sym}^{2m}\mathbb{C}^2$. Let $V_{\mathbb{R}}(m)$ be $V(m)$ regarded as a real vector space and let $\rho_{\mathbb{R}}(m)$ be the representation $\rho(m)$ regarded as a real representation. There exists a canonical lattice $L(m)$ in $V_{\mathbb{R}}(m)$ which is invariant under $\rho_{\mathbb{R}}(m)(\Gamma(D))$, see section 4. Then $L(m)$ can be regarded as an arithmetic Γ_0-module. Actually, we work with a slightly symmetrized version of these lattices. Let $L^*(m)$ be the dual lattice of $L(m)$ in $V_{\mathbb{R}}^*(m)$. Then we consider the lattice $\overline{L}(m) := L(m) \oplus L^*(m)$ in the space $V_{\mathbb{R}}(m) \oplus V_{\mathbb{R}}^*(m)$ on which $\mathrm{SL}_2(\mathbb{C})$ acts with the representation $\rho_{\mathbb{R}}(m)_{\mathbb{R}} \oplus \check{\rho}_{\mathbb{R}}(m)$, where $\check{\rho}_{\mathbb{R}}(m)$ denotes the contragredient representation of $\rho_{\mathbb{R}}(m)$, which in our case is in fact equivalent to $\rho_{\mathbb{R}}(m)$.

The main result of this article is that the size of the torsion part in the first integral homology with coefficients in $\overline{L}(m)$ grows exponentially for certain sequences of congruence subgroups of Bianchi groups. More precisely, we will prove the following theorem.

Theorem 1.1. *Let $\Gamma_0 \subset \Gamma(D)$ be a congruence subgroup and assume that Γ_0 is torsion-free. Let Γ_i, $i \in \mathbb{N}$ be a sequence of congruence subgroups of $\Gamma(D)$, contained in Γ_0, with*

$$\lim_{i\to\infty} \frac{\kappa(\Gamma_i)\log[\Gamma(D) : \Gamma_i]}{[\Gamma(D) : \Gamma_i]} = 0, \tag{1.3}$$

where $\kappa(\Gamma_i)$ *denotes the number of cusps of* Γ_i. *Let* $X_0 := \Gamma_0\backslash\mathbb{H}^3$. *Then for* $m \in \mathbb{N}$ *with* $m \geq 3$ *one has*

$$\liminf_{i\to\infty} \frac{\log|H_1(\Gamma_i : \overline{L}(m))_{tors}|}{[\Gamma_0 : \Gamma_i]} \geq \frac{2m(m+1)-12}{\pi}\operatorname{vol}(X_0). \tag{1.4}$$

Our main result Theorem 1.1 can be applied for example to the standard exhaustive sequences of congruence subgroups. Namely, the following corollary holds.

Corollary 1.2. *If* $\mathfrak{a}_i$ *is a chain of ideals in* $\mathcal{O}_D$ *such that* $\lim_{i\to\infty} N(\mathfrak{a}_i) = \infty$, *then for* $m \geq 3$ *one has*

$$\liminf_{i\to\infty} \frac{\log|H_1(\Gamma(\mathfrak{a}_i); \overline{L}(m))_{tors}|}{\operatorname{vol}(\Gamma(\mathfrak{a}_i)\backslash\mathbb{H}^3)} \geq \frac{2m(m+1)-12}{\pi}.$$

In particular, for any congruence subgroup Γ_0 *of* $\Gamma(D)$ *there exists a sequence* $\Gamma_0 \supset \Gamma_1 \supset \dots$ *with* $[\Gamma_0 : \Gamma_i] < \infty$, $\cap_i\Gamma_i = 1$ *and which satisfies* (1.4).

We shall now briefly describe our approach to prove our main results. We let Γ be a congruence subgroup of $\Gamma(D)$ which is torsion-free. Let $\mathbb{H}^3$ be the hyperbolic 3-space. Then $X := \Gamma\backslash\mathbb{H}^3$ is a hyperbolic manifold with cusps of finite volume. As already observerd by Bergeron and Venkatesh, the size of the torsion subgroups $H_q(\Gamma, L(m))_{tors}$ is closely related to the Reidemeister torsion of X with coefficients in the complex flat vector bundle $E(\rho(m))$ defined by the restriction of $\rho(m)$ to Γ. However, in our case the singular homology of this bundle never vanishes. This is a major difference to the cocompact situation and causes additional difficulties: The Reidemeister torsion with coefficients in $E(\rho(m))$ is not an invariant of X and $\rho(m)$ but it depends on the choice of a particular basis the in the singular homology. To overcome this problem, Menal-Ferrer and Porti introduced the so called normalized Reidemeister torsion of X with coefficients in $E(\rho(m))$, see [MePo2]. This torsion is defined as a quotient of two Reidemeister torsions for the representations $\rho(m)$ and $\rho(2)$, $m \geq 3$ with respect to bases in the singular homology coming from the boundary of the Borel-Serre compactification of X and its definition is independent of the particular choice of such bases, see [MePo2, Proposition 2.2]. The choice of a particular basis also enters the relation between the Reidemeister torsion and the size of the torsion subgroups in terms of certain volume factors in the free part of the homology.

To prove our main results, in a first step we show that the normalized Reidemeister torsion grows exponentially in certain sequences of coverings. The basic tool we use is an expression of the normalized Reidemeister torsion in terms of special values of Ruelle zeta functions which has been proved by Menal-Ferrer and Porti [MePo2, Theorem 5.8], who generalized a result of Müller [Mü2, equation 8.7] to the non-compact case.

In a second step, expressing the normalized Reidemeister torsion as a weighted product of sizes of the groups $|H_q(\Gamma, \overline{L}(m))_{tors}|$, $|H_q(\Gamma, \overline{L}(2)_{tors}|$ and a contribution from the free part in the singular homology, we show that the latter contribution can be controlled in the limit and that the contribution of $|H_1(\Gamma, \overline{L}(m))_{tors}|$ can be isolated in such a way that Theorem 1.1 can be deduced. This step covers the main part of our proof.

The structure of this paper is as follows. In section 2 we recall the definition of the normalized Reidemeister torsion and prove our main result about its asymptotic behaviour. In section 3 we recapitulate some facts about the homology of arithmetic groups with coefficients in free $\mathbb{Z}$-modules. In section 4 we introduce the Bianchi groups and their congruence subgroups. The proof of Theorem 1.1 and Corollary 1.2 will be carried out in the last section 5.

At the end of this introduction we want to remark that in parts of his thesis [Ra1] and in his recent preprint [Ra2] Jean Raimbault studied the asymptotic behaviour of the regularized analytic torsion and of the Reidemeister torsion for more general sequences of hyperbolic 3-manifolds and for more general strongly acyclic coefficient systems. This problem is closely related to the problem studied in this paper.

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2. Ruelle zeta functions and the asymptotics of normalized Reidemeister torsion

Let X be an oriented hyperbolic 3-manifold of finite volume with a fixed spin-structure. Then there exists a discrete, torsion-free subgroup Γ of $\mathrm{SL}_2(\mathbb{C})$ such that $X = \Gamma\backslash\mathbb{H}^3$. We assume from now on that X is not compact. Then X is of the form

$$X = C(X) \cup \bigsqcup_{i=1}^{\kappa(X)} F_i,$$

with $C(X) \cap F_i = \partial C(X) \cap \partial F_i$ and $F_i \cap F_j = \emptyset$ for $i \neq j$. Here $C(X)$ is a compact smooth manifold with boundary $\partial C(X)$ and each F_i, a cusp of X, is diffeomorphic to $[1,\infty) \times T^2$, T^2 the 2-dimensional torus. Moreover, on each F_i the hyperbolic metric of X restricts to the warped product metric $y^{-2}dy^2 + y^{-2}g_0$, where g_0 denotes the suitably normalized flat metric on T^2. Thus, gluing at 2-torus to each end of X we obtain a compact smooth manifold $\overline{X}$ with boundary such that X is the interior of $\overline{X}$ and such that $\overline{X}$ is homotopy-equivalent to X.

Let $\mathcal{C}(X)$ be the set of non-trivial closed geodesics of X. For $c \in \mathcal{C}(X)$ let $\ell(c)$ denote its length. Then there exists a constant c_X such that for each $R \in (0,\infty)$ one has

$$\#\{c \in \mathcal{C}(X) \colon \ell(c) \leq R\} \leq c_X e^{2R}, \tag{2.5}$$

see for example [MePo2, Lemma 4.3, Proposition 4.4]. We let

$$\ell(X) := \ell(\Gamma) := \min\{\ell(c) \colon c \in \mathcal{C}(X)\}$$

A closed geodesic $c \in \mathcal{C}(X)$ is called prime if it is the shortest among the closed geodesics having the same image as c. The set of prime geodesics will be denoted by $\mathcal{PC}(X)$. If $c \in \mathcal{C}(X)$, there exists a unique prime geodesic $c_0 \in \mathcal{C}(X)$ with the same image as c and thus there exists a unique integer $n(c)$ such that $\ell(c) = n(c)\ell(c_0)$.

For $\gamma \in \Gamma$ we let $[\gamma]$ be its conjugacy class in Γ. We let $C(\Gamma)_s$ be the set of all conjugacy classes $[\gamma]$ such that $\gamma \in \Gamma$ is semisimple and non-trivial. Then the set $C(\Gamma)_s$ corresponds bijectively to $\mathcal{C}(X)$, see for example [Pf, section 3]. If $\gamma \in \Gamma$ is semisimple, we let $c(\gamma)$ be the closed geodesic associated to $[\gamma]$ and we let $\ell(\gamma)$ be its length. Let

$$M := \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in [0, 2\pi) \right\}; \quad H_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then there exists a unique $m_\gamma \in M$ such that γ is G-conjugate to $m_\gamma \exp(\ell(\gamma) H_1)$, see for example [Pf, section 3]. For $k \in \frac{1}{2}\mathbb{Z}$ we let σ_k be the one-dimensional representation of M on $\mathbb{C}(\sigma_k) := \mathbb{C}$ defined by

$$\sigma_k \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} := e^{2ik\theta}.$$

Now we let SX be the unit-sphere bundle of X. Then there is a canonical isomorphism

$$SX \cong \Gamma \backslash G / M.$$

Thus every $\sigma = \sigma_k$ defines a locally homogeneous vector bundle

$$V(\sigma) := \Gamma \backslash G \times_\sigma \mathbb{C}(\sigma)$$

over SX. Moreover, the geodesic flow Φ of SX lifts to a flow $\Phi(\sigma)$ of $V(\sigma)$. Thus for every closed geodesic c its lift to $V(\sigma)$ defines an endomorphism $\mu_\sigma(c)$ on the fibre of $V(\sigma)$ over $\dot{c}(0)$. Explicitly, if one regards μ_σ as an endomorphism on $\mathbb{C}(\sigma)$, one has $\mu_\sigma(c(\gamma)) = \sigma(m_\gamma)$ for every semisimple element $\gamma \in \Gamma$. Now for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 2$, the Ruelle zeta function of X associated to the representation σ is defined by

$$R_X(s, \sigma) := \prod_{c \in \mathcal{PC}(X)} \det \left(\mathrm{Id} - \mu_\sigma(c) e^{-s\ell(c)} \right). \tag{2.6}$$

To establish the convergence of this product, we remarkt that one has

$$\begin{aligned} \log R_X(s, \sigma) &= \sum_{c \in \mathcal{PC}(X)} \log \left(1 - \mu_\sigma(c) e^{-s\ell(c)} \right) \\ &= - \sum_{c \in \mathcal{PC}(X)} \sum_{k=1}^{\infty} \frac{(\mu_\sigma(c))^k e^{-ks\ell(c)}}{k} \\ &= - \sum_{c \in \mathcal{C}(X)} \frac{\mu_\sigma(c) e^{-s\ell(c)}}{n(c)}. \end{aligned} \tag{2.7}$$

Thus by (2.5), the last series converges absolutely for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 2$. We remark that if X is compact or if more generally X is of finite volume and Γ is neat, then by [BO], [Pf] the function $R(s, \sigma)$ admits a meromorphic continuation to $\mathbb{C}$. Under coverings, the Ruelle zeta functions can be estimated as follows.

Lemma 2.1. *Let X_0 be an oriented hyperbolic 3-manifold of finite volume. Then there exists a constant $C(X_0)$ such that for each hyperbolic 3-manifold X which is a finite covering of X_0 of index $[X_0 : X]$ and every $s \in [3, \infty)$ one can estimate*

$$|\log R_X(s, \sigma_k)| \leq C(X_0)[X_0 : X]e^{-\frac{\ell(X)}{2}}$$

Proof. Let $\pi : X \to X_0$ be a covering of index $[X_0 : X] < \infty$. Let c be a closed non-trivial geodesic in X of length $\ell(c)$. Then $\pi(c)$ is a closed non-trivial geodesic in X_0 of the same length. Moreover, there are at most $[X_0 : X]$ closed geodesics c' in X projecting to $\pi(c)$. Thus by (2.7), for $s \in [3, \infty)$ one can estimate

$$|\log R_X(s, \sigma_k)| \leq e^{-\frac{\ell(X)}{2}} \sum_{c \in \mathcal{C}(X)} e^{-(s-\frac{1}{2})\ell(c)} \leq e^{-\frac{\ell(X)}{2}}[X_0 : X] \sum_{c \in \mathcal{C}(X_0)} e^{-(s-\frac{1}{2})\ell(c)}.$$

Thus if we let

$$C(X_0) := \sum_{c \in \mathcal{C}(X_0)} e^{-\frac{5}{2}\ell(c)},$$

then $C(X_0) < \infty$ by (2.5) and the lemma follows. □

Since X is homotopy-equivalent to $\overline{X}$, we can identify Γ with the fundamental group of $\overline{X}$. Restricting $\rho(m)$ to Γ, we can form the homology groups $H_q(\overline{X}; V(m))$ with coefficients in the complex Γ-module $V(m)$, see section 3. These groups are finite-dimensional complex vector spaces. We assume from now on that $m \geq 1$.

Let $\partial\overline{X}$ be the boundary of $\overline{X}$, let T_j, $j = 1 \dots, \kappa(X)$ be the components of $\partial\overline{X}$ and for each j let $\iota : T_j \to \overline{X}$ denote the inclusion. Then we obtain a map from of the fundamental group $\pi_1(T_j)$ of T_j into Γ and in this way we regard $V(m)$ as a $\pi_1(T_j)$-module. Then the corresponding homology groups with coefficients will be denoted by $H_q(T_j, V(m))$. For each q there is a natural map $\iota_* : H_q(T_j; V(m)) \to H_q(\overline{X}; V(m))$.

According to Menal-Ferrer and Porti, [MePo2], this map can be used to define bases in the homology groups $H_1(\overline{X}; V(m))$, $H_2(\overline{X}; V(m))$ as follows. Fix for each j a non-zero vector $\omega_j(m) \in V(m)$ fixed under $\rho(m)(\pi_1(T_j))$. The set of such vectors is a one-dimensional complex vector space, see [MePo1, Lemma 2.4]. Fix moreover for each j a non-trivial cylce $\theta_j \in H_1(T_j; \mathbb{Z})$ and let $\eta_j \in H_2(T_j; \mathbb{Z})$ be a $\mathbb{Z}$-generator of $H_2(T_j; \mathbb{Z})$. Then the elements $\theta_j \otimes \omega_j(m)$ resp. $\eta_j \otimes \omega_j(m)$ are elements of $H_1(T_j, V(m))$ resp. $H_2(T_j, V(m))$ and the following proposition holds.

Proposition 2.2. *The maps $\iota_* : H_2(\partial\overline{X}; V(m)) \to H_2(\overline{X}; V(m))$ and $\iota_* : H_1(\partial\overline{X}; V(m)) \to H_1(\overline{X}; V(m))$ are surjective. More precisely, let $\theta := \{\theta_1, \dots, \theta_{\kappa(X)}\}$, let $\eta := \{\eta_1, \dots, \eta_{\kappa(X)}\}$, and let $\omega(m) := \{\omega_1(m), \dots, \omega_{\kappa(X)}(m)\}$. Then the sets*

$$\mathcal{B}(\theta; \omega(m)) := \bigsqcup_{j=1}^{\kappa(X)} \{\iota_*(\theta_j \otimes \omega_j(m))\};\ \mathcal{B}(\eta; \omega(m)) := \bigsqcup_{j=1}^{\kappa(X)} \{\iota_*(\eta_j \otimes \omega_j(m))\} \tag{2.8}$$

form a basis of $H_1(\overline{X}; V(m))$ resp. $H_2(\overline{X}; V(m))$.

Proof. This is proved by Menal-Ferrer and Porti, [MePo2, Proposition 2.10]. □

Since $\overline{X}$ is homotopy-equivalent to a 2-dimensional CW-complex, the group $H_3(\overline{X}; V(m))$ is trival. Since $V(m)$ is self-contragredient, one has $H_0(\overline{X}; V(m) \cong H^0(\overline{X}; V(m))$ and since the last group coincides with the Γ-invariant vectors in $V(m)$, $H_0(\overline{X}; V(m))$ is also trivial. Thus $\mathcal{B}(\theta, \eta, \omega(m)) := \mathcal{B}(\theta; \omega(m)) \sqcup \mathcal{B}(\eta; \omega(m))$ defines a basis in the homology of $\overline{X}$ with respect to the local system associated to $\rho(m)$.

Let $T_{\overline{X}}(\rho(m); \mathcal{B}(\theta, \eta, \omega(m))$ be the Reidemeister-torsion of $\overline{X}$ associated to the restriction of the complex representation $\rho(m)$ to Γ and with respect to the basis $\mathcal{B}(\theta, \eta, \omega(m))$, see [Mü1, section 1]. Then by [MePo2, Proposition 2.2], for $m \geq 3$ the quotient

$$\mathcal{T}_{\overline{X}}(\rho(m)) := \frac{T_{\overline{X}}(\rho(m); \mathcal{B}(\theta, \eta, \omega(m))}{T_{\overline{X}}(\rho(2); \mathcal{B}(\theta, \eta, \omega(2)))}$$

is independent of the choice of θ. It is easy to see that the quotient is also independent of $\omega(m)$, $\omega(2)$ and η. We remark that since $\rho(m)$ factors through a representation of $\mathrm{PSL}_2(\mathbb{C})$, $T_{\overline{X}}(\rho(m))$ is also independent of the spin-structure of X. This independence of the spin-structure is the main reason why we restrict to even symmetric powers of the standard representation in this article.

We shall now study the asymptotic behaviour of the normalized Reidemeister torsion under sequences of hyperbolic manifolds X_i. If the constant $c_{\mathbb{H}^3}(\rho(m))$ is defined according to (1.2), one has

$$c_{\mathbb{H}^3}(\rho(m)) = \frac{m(m+1)}{\pi} + \frac{1}{6\pi}, \tag{2.9}$$

see [BV, 5.9.3, Example 3], [MüPf, Remark 2]. Moreover, the following proposition holds.

Proposition 2.3. *Let X_0 be an oriented hyperbolic 3-manifold of finite volume. Let X_i, $i \in \mathbb{N}$ be a sequence of hyperbolic 3-manifolds which are finite coverings of X_0 such that $\lim_{i\to\infty} \ell(X_i) = \infty$. Then for every $m \in \mathbb{N}$ with $m \geq 3$ one has*

$$\lim_{i\to\infty} \frac{\log \mathcal{T}_{\overline{X}_i}(\rho(m))}{\mathrm{vol}(X_i)} = -(c_{\mathbb{H}^3}(\rho(m)) - c_{\mathbb{H}^3}(\rho(2))) = -\frac{m(m+1)-6}{\pi}.$$

Proof. By [MePo2, Theorem 5.8], taking the different parametrizations into account, we have

$$\log \mathcal{T}_{\overline{X}_i}(\rho(m)) = -\frac{1}{\pi} \mathrm{vol}\,(X_i)(m(m+1)-6) + \sum_{k=3}^{m} \log |R_{X_i}(k, \sigma_k)|\,.$$

Applying Lemma 2.1, the proposition follows. □

To explain Proposition 2.3 we note that for $m \in \mathbb{N} - \{0\}$ the representation $\rho(m)$ is not invariant under the standard Cartan-Involution of $\mathrm{SL}_2(\mathbb{C})$. Thus by [BV, Lemma 4.1], for closed hyperbolic 3-manifolds X the bundle $E(\rho(m))$ over X is strongly acyclic. In particular, by the Hodge-DeRham isomorphism, in the closed case the homology groups $H_q(X, E(\rho(m)))$ vanish and the Reidemeister torsion $T_X(\rho(m))$ is an invariant of the closed manifold X and the representation $\rho(m)$. Moreover, as a special case of a more general

Theorem of Bergeron and Venkatesh [BV, Theorem 4.5] one obtains for a sequence of closed hyperbolic 3-manifolds X_i with $\ell(X_i) \to \infty$ and for every $m \in \mathbb{N} - \{0\}$ that

$$\lim_{i\to\infty} \frac{\log T_{X_i}(\rho(m))}{\operatorname{vol}(X_i)} = -c_{\mathbb{H}^3}(\rho(m)).$$

Since in the closed case one has $\log \mathcal{T}_{X_i}(m) = \log T_{X_i}(\rho(m)) - \log T_{X_i}(\rho(2))$, Proposition 2.3 can be seen as a modified extension of the result of Bergeron and Venkatesh to the non-compact 3-dimensional case.

3. Torsion in the homology of arithmetic groups

We keep the notation of the previous section. We fix a finite-dimensional smooth representation ρ of $\mathrm{SL}_2(\mathbb{C})$ on a complex vector-space V. We let $V_{\mathbb{R}}$ be V, regarded as a real vector-space, and we we let $\rho_{\mathbb{R}}$ be the corresponding real representation of $\mathrm{SL}_2(\mathbb{C})$. Assume that there exists a lattice L in $V_{\mathbb{R}}$ which is invariant under $\rho_{\mathbb{R}}(\Gamma)$. Then we obtain a representation $\rho_{\mathbb{Z}} : \Gamma \to \mathrm{Aut}_{\mathbb{Z}}(L)$. For $q = 0, 1, 2$ we denote the associated homology resp. cohomology groups of Γ with coefficients in L by $H_q(\Gamma; L)$ resp. $H^q(\Gamma; L)$. These groups can be computed as follows. Let K be a smooth triangulation of $\overline{X}$, containing a subcomplex J triangulating $\partial\overline{X}$. Let $\tilde{X}$ be the universal covering of $\overline{X}$. Then $\tilde{X}$ is homotopy equivalent to $\mathbb{H}^3$, the hyperbolic 3-space. In particular, $\tilde{X}$ is contractible. Let $C_q(\tilde{K})$ be the free abelian group generated by the q-chains of $\tilde{K}$, let $C^q(\tilde{K}) := \mathrm{Hom}_{\mathbb{Z}}(C_q(\tilde{K}), \mathbb{Z})$ and let $C_*(\tilde{K})$ resp. $C^*(\tilde{K})$ be the associated simplical chain- resp. cochain-complexes. Each $C_q(\tilde{K})$ is a free $\mathbb{Z}[\Gamma]$ module and if one fixes an embedding of K into $\tilde{K}$, then the q-cells of K form a basis of $C_q(\tilde{K})$ over $\mathbb{Z}[\Gamma]$. Let

$$C_q(K; L) := C_q(\tilde{K}) \otimes_{\mathbb{Z}[\Gamma]} L; \quad C^q(K; L) := C^q(\tilde{K}) \otimes_{\mathbb{Z}[\Gamma]} L.$$

Then the $C_q(K; L)$ resp. $C^q(K; L)$ again form a chain resp. a cochain complex $C_*(K; L)$ resp. $C^*(K; L)$ of free $\mathbb{Z}$-modules of finite rank and the corresponding homology resp. cohomology groups, which are topological invariants of $\overline{X}$, will be denoted by $H_q(\overline{X}; L)$ resp. $H^q(\overline{X}; L)$. Moreover, since $\tilde{X}$ is contractible, the $C_q(\tilde{K})$ form a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[\Gamma]$ and thus one has isomorphisms

$$H_q(\Gamma; L) \cong H_q(\overline{X}; L); \quad H^q(\Gamma; L) \cong H^q(\overline{X}; L),$$

where the second isomorphism follows from the isomorphism

$$C^q(K, L) \cong \mathrm{Hom}_{\mathbb{Z}[\Gamma]}(C_q(\tilde{K}), L),$$

which induces an isomorphism of the corresponding cochain complexes.

Let $\tilde{J}$ be the subcomplex of $\tilde{K}$ coming from the subcomplex J. Restricting $\rho_{\mathbb{Z}}$ to the image of $\pi_1(\partial\overline{X})$ in Γ and using $\tilde{J}$, we can form the complex $C_*(J; L))$ which is a subcomplex of $C_*(K; L)$. The homology groups of $C_*(J; L)$ are topological invariants of $\partial\overline{X}$ and will be denoted by $H_q(\partial\overline{X}; L)$. Finall by $H_q(\overline{X}, \partial\overline{X}; L)$ we denote the relative homology, i.e. the homology of the complex $C_*(K; L)/C_*(J; L)$.

If we denote by A one of the homology resp. cohomology groups, then A is a finitely generated abelian group and thus it has a decomposition $A = A_{free} \oplus A_{tors}$, where A_{free}

is a finite-rank free $\mathbb{Z}$-module and where A_{tors} is a finite abelian group. Now we let $L^* := \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$. Then L^* becomes a Γ-module via the contragredient representation $\check{\rho}_{\mathbb{Z}}$ of $\rho_{\mathbb{Z}}$. Moreover, we have the following Lemma.

Lemma 3.1. *For each q there is an isomorphism $H^q(\Gamma; L^*)_{tors} \cong H_{q-1}(\Gamma; L))_{tors}$.*

Proof. There is an isomorphism of complexes

$$C^*(K; L^*) \cong \mathrm{Hom}_{\mathbb{Z}}(C_*(K; L); \mathbb{Z})$$

and thus the statement follows from the universal coefficient theorem. □

Now we let

$$C_q(K, V_{\mathbb{R}}) := C_q(\tilde{K}) \otimes_{\mathbb{R}[\Gamma]} V_{\mathbb{R}}.$$

The $C_q(K, V_{\mathbb{R}})$ form a chain complex $C_*(\tilde{K}, V_{\mathbb{R}})$ of finite-dimensional $\mathbb{R}$-vector spaces and

$$C_q(K, V_{\mathbb{R}}) = C_q(K, L) \otimes_{\mathbb{Z}} \mathbb{R}. \tag{3.10}$$

The homology groups $H_q(K, V_{\mathbb{R}})$ of the complex $C_*(\tilde{K}, V_{\mathbb{R}})$ are topological invariants of $\overline{X}$ and are equal to the homology groups $H_q(\overline{X}; V_{\mathbb{R}})$ of $\overline{X}$ with coefficients in the local system defined by $\rho_{\mathbb{R}}$. By (3.10), $H_q(\overline{X}; L)_{free}$ is a lattice in $H_q(\overline{X}; V_{\mathbb{R}})$.

Similarly, if we regard ρ as a complex representation of Γ, we obtain the cohomology groups $H_q(\overline{X}; V)$, which are complex vector spaces. Regarded as real vector spaces, they are equal to the $H_q(\overline{X}; V_{\mathbb{R}})$.

Now assume that we are given bases $\mathcal{B}_q^{\mathbb{R}}$ of $H_q(\overline{X}; V_{\mathbb{R}})$, $q \in \{1, 2\}$. Let $\mathcal{B}^{\mathbb{R}} = \mathcal{B}_1^{\mathbb{R}} \sqcup \mathcal{B}_2^{\mathbb{R}}$. We define an inner product on $H_q(\overline{X}; V_{\mathbb{R}})$ by declaring $\mathcal{B}_q^{\mathbb{R}}$ to be an orthonormal basis. By $\mathrm{vol}_{\mathcal{B}_q^{\mathbb{R}}} H_q(\overline{X}; L)_{free}$ we denote the covolume of the lattice $H_q(\overline{X}; L)_{free}$ with respect to this inner product. Then if $T_{\overline{X}}(\rho_{\mathbb{R}}; \mathcal{B}^{\mathbb{R}})$ is the Reidemeister torsion of $\overline{X}$ associated to this inner product and the local coefficients defined by $\rho_{\mathbb{R}}|_{\Gamma}$, the following lemma holds.

Lemma 3.2. *One has*

$$T_{\overline{X}}(\rho_{\mathbb{R}}; \mathcal{B}^{\mathbb{R}}) = \frac{|H_0(\Gamma; L)_{tors}| \, \mathrm{vol}_{\mathcal{B}_1^{\mathbb{R}}} H_1(\overline{X}; L)_{free}}{|H_1(\Gamma; L)_{tors}| \, \mathrm{vol}_{\mathcal{B}_2^{\mathbb{R}}} H_2(\overline{X}; L)_{free}}.$$

Proof. This is proved by Bergeron and Venkatesh [BV, section 2.2]. □

We finally have to relate the Reidemeister torsion of the representation $\rho_{\mathbb{R}}$ to the Reidemeister torsion of the representation ρ, regarded as a complex representation. If $\mathcal{B}_q$, $q \in \{1, 2\}$ are bases of $H_q(\overline{X}; V)$, $\mathcal{B} = \mathcal{B}_1 \sqcup \mathcal{B}_2$, we define bases $\mathcal{B}_q^{\mathbb{R}}$ of $H_q(\overline{X}; V_{\mathbb{R}})$ by

$$\mathcal{B}_q^{\mathbb{R}} := \mathcal{B}_q \sqcup \{\sqrt{-1}\omega \colon \omega \in \mathcal{B}_q\}; \quad \mathcal{B}^{\mathbb{R}} := \mathcal{B}_1^{\mathbb{R}} \sqcup \mathcal{B}_2^{\mathbb{R}}. \tag{3.11}$$

Then the following Lemma holds.

Lemma 3.3. *For any complex basis $\mathcal{B}$ as above one has*

$$T_X(\rho; \mathcal{B})^2 = T_X(\rho_{\mathbb{R}}; \mathcal{B}^{\mathbb{R}}).$$

Proof. For the proof one can proceed as in the proof of [MaMü, Lemma 2.4]. □

4. Congruence subgroups of Bianchi groups

In this section we collect some basic properties of the Bianchi groups and their congruence subgroups which are needed for our purposes. Let us firstly recall the definition of these groups. We let $F := \mathbb{Q}(\sqrt{-D})$, $D \in \mathbb{N}$ square-free, be an imaginary quadratic number field and d_F be its class number. Let $\mathcal{O}_D$ be the ring of integers of F, i.e. $\mathcal{O}_D = \mathbb{Z} + \sqrt{-D}\mathbb{Z}$ if $D \equiv 1, 2$ modulo 4, $\mathcal{O}_D = \mathbb{Z} + \frac{1+\sqrt{-D}}{2}\mathbb{Z}$ if $D \equiv 3$ modulo 4. We let $\Gamma(D) := \mathrm{SL}_2(\mathcal{O}_D)$ be the associated Bianchi-group. Then $X_D := \Gamma(D)\backslash\mathbb{H}^3$ is of finite volume

$$\mathrm{vol}(X_D) = \frac{|\delta_F|^{\frac{3}{2}}\zeta_F(2)}{4\pi^2},$$

where ζ_F is the Dedekind zeta function of F and δ_F is is the discriminant of F, see [Hu], [Sa, Proposition 2.1]. Let $\mathfrak{a}$ be any nonzero ideal in $\mathcal{O}_D$ and let $N(\mathfrak{a})$ denote its norm. Then the associated principal congruence subgroup $\Gamma(\mathfrak{a})$ is defined as

$$\Gamma(\mathfrak{a}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_D) \colon a-1 \in \mathfrak{a}; d-1 \in \mathfrak{a}; b, c \in \mathfrak{a} \right\}.$$

A subgroup Γ of $\Gamma(D)$ is called a congruence subgroup if there exists a non-zero ideal $\mathfrak{a}$ in $\mathcal{O}_D$ such that Γ contains $\Gamma(\mathfrak{a})$ as a subgroup of finite index. We recall that by [Ba, Corollary 5.2] the sequence

$$1 \to \Gamma(\mathfrak{a}) \to \Gamma(D) \to \mathrm{SL}_2(\mathcal{O}_D/\mathfrak{a}) \to 1$$

is exact. Thus, arguing exactly as in [Sh, Chapter 1.6] for the $\mathrm{SL}_2(\mathbb{R})$-case, one obtains

$$[\Gamma(D) : \Gamma(\mathfrak{a})] = \#\,\mathrm{SL}_2(\mathcal{O}_D/\mathfrak{a}) = N(\mathfrak{a})^3 \prod_{\mathfrak{p}|\mathfrak{a}} \left(1 - \frac{1}{N(\mathfrak{p})^2}\right), \tag{4.12}$$

where the product is taken over all prime ideals $\mathfrak{p}$ in $\mathcal{O}_D$ dividing $\mathfrak{a}$. Let $\mathbb{P}^1(F)$ be the one-dimensional projective space of F. As usual, we write ∞ for the element $[1, 0] \in \mathbb{P}^1(F)$. Then $\mathrm{SL}_2(F)$ acts naturally on $\mathbb{P}^1(F)$ and by [EGM, Chapter 7.2, Proposition 2.2] one has $\kappa(\Gamma(D)) = \#\,(\Gamma(D)\backslash\mathbb{P}^1(F))$, $\kappa(\Gamma(\mathfrak{a})) = \#\,(\Gamma(\mathfrak{a})\backslash\mathbb{P}^1(F))$. Furthermore, by [EGM, Chapter 7.2, Theorem 2.4] one has $\kappa(\Gamma(D)) = d_F$. Let $P = MAN$ be the standard parabolic subgroup of $\mathrm{SL}_2(\mathbb{C})$, where M is as above and where

$$A = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda \in \mathbb{R}, \lambda > 0 \right\}; \quad N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, b \in \mathbb{C} \right\}.$$

Then P is the stabilizer of ∞ in $\mathrm{SL}_2(\mathbb{C})$. For each $\eta \in \mathbb{P}^1(F)$ we fix $B_\eta \in \mathrm{SL}_2(F)$ with $B_\eta\eta = \infty$. Then $P_\eta := B_\eta^{-1}PB_\eta$ is the stabilizer of η in $\mathrm{SL}_2(\mathbb{C})$. We let $N_\eta := B_\eta^{-1}NB_\eta$. If $\eta \in \mathbb{P}^1(F)$, we let $\Gamma(D)_\eta$ resp. $\Gamma(\mathfrak{a})_\eta$ be the stabilizer of η in $\Gamma(D)$ resp. $\Gamma(\mathfrak{a})$.

The next Lemma is certainly well known to experts. However, since we could not find a reference, we include a proof here. We let $\mathcal{O}_D^*$ be the group of units of $\mathcal{O}_D$, i.e. $\mathcal{O}_D^* = \{\pm 1\}$ for $D \neq 1, 3$, $\mathcal{O}_D^* = \{\pm 1, \pm\sqrt{-1}\}$ for $D = 1$, $\mathcal{O}_D^* = \{\pm 1, \pm\frac{1\pm\sqrt{-3}}{2}\}$ for $D = 3$.

Lemma 4.1. *Let $\mathfrak{a}$ be an ideal in $\mathcal{O}_D$. Then for $N(\mathfrak{a})$ sufficiently large one has* $\kappa(\Gamma(\mathfrak{a})) = d_F \frac{[\Gamma(D):\Gamma(\mathfrak{a})]}{\#(\mathcal{O}_D^*)N(\mathfrak{a})}$.

Proof. The group P is defined over F and we let $P(F)$ be its F-valued points. If $\lambda \in \mathbb{C}$ is an eigenvalue of an element $\gamma \in \Gamma(D)_\eta$, then, since $B_\eta \gamma B_\eta^{-1} \subset P(F)$, also λ^{-1} is an eigenvalue and thus one has $\lambda, \lambda^{-1} \in F$ and since $\mathcal{O}_D$ is integrally closed, one has $\lambda \in \mathcal{O}_D^*$. Moreover, if $\mathcal{O}_D^* \neq \{\pm 1\}$, then $D = 1, 3$ and in this case the class number is one and so $\mathcal{O}_D$ has only one cusp, so in this case one can assume that $B_\eta \in \mathrm{SL}_2(\mathcal{O}_D)$. Thus in any case one obtains

$$B_\eta \Gamma(D)_\eta B_\eta^{-1} = J(B_\eta \Gamma(D)_\eta B_\eta^{-1} \cap N),\ J \in \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \alpha \in \mathcal{O}_D^* \right\}. \tag{4.13}$$

Assume that $D \neq 1, 3$. Then, since for every element $\gamma \in \Gamma(\mathfrak{a})$ one has $\mathrm{Tr}(\gamma) \in 2 + \mathfrak{a}$ and since $\mathrm{Tr}(g) = -2$ for every element $g \in -N$, it follows that for $-4 \notin \mathfrak{a}$, i.e. for $N(\mathfrak{a})$ sufficiently large, one has $B_\eta \Gamma(\mathfrak{a})_\eta B_\eta^{-1} \subset N$. If $D = 1, 3$, then since $\Gamma(\mathfrak{a})$ is a normal subgroup of $\Gamma(D)$ one has $B_\eta \Gamma(\mathfrak{a})_\eta B_\eta^{-1} = \Gamma(\mathfrak{a})_\infty$ and it follows from (4.13) that $\Gamma(\mathfrak{a})_\infty \subset N$ for $N(\mathfrak{a})$ sufficiently large.

Now for $\eta \in \mathbb{P}^1(F)$, B_η as above, write $B_\eta = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(F)$ and let $\mathfrak{u}$ be the $\mathcal{O}_D$-module generated by γ and δ. Then one has

$$B_\eta \Gamma(D)_\eta B_\eta^{-1} \cap N = \left\{ \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}; \omega \in \mathfrak{u}^{-2} \right\};\ B_\eta \Gamma(\mathfrak{a})_\eta B_\eta^{-1} \cap N = \left\{ \begin{pmatrix} 1 & \omega' \\ 0 & 1 \end{pmatrix}; \omega' \in \mathfrak{a}\mathfrak{u}^{-2} \right\},$$

where the first equality is proved in [EGM, Chapter 8.2, Lemma 2.2] and where the second equality can be proved using the same arguments. Thus one has

$$[B_\eta \Gamma(D)_\eta B_\eta^{-1} \cap N : B_\eta \Gamma(\mathfrak{a})_\eta B_\eta^{-1} \cap N] = N(\mathfrak{a}).$$

Thus by (4.13), for each $\eta \in \mathbb{P}^1(F)$ and $N(\mathfrak{a})$ suffciently large one has $[\Gamma(D)_\eta : \Gamma(\mathfrak{a})_\eta] = \#(\mathcal{O}_D^*)N(\mathfrak{a})$ and so, if $\eta_1, \dots, \eta_{d_F}$ denote fixed representatives of $\Gamma(D)\backslash\mathbb{P}^1(F)$ one obtains

$$\kappa(\Gamma(\mathfrak{a})) = \#(\Gamma(\mathfrak{a})\backslash\mathbb{P}^1(F)) = \#\left(\bigsqcup_{i=1}^{d_F} \Gamma(\mathfrak{a})\backslash\Gamma(D)/\Gamma(D)_{\eta_i} \right) = d_F \frac{[\Gamma(D) : \Gamma(\mathfrak{a})]}{\#(\mathcal{O}_D^*)N(\mathfrak{a})}.$$

□

We shall now describe the lattices $L(m)$ in $V_\mathbb{R}(m)$. Let e_1, e_2 be the standard basis of $\mathbb{C}^2$. Then if we realize $V(m)$ as the space of homogeneous polynomials in e_1 and e_2 of degree $2m$, a complex basis of $V(m)$ is given by $\{e_1^{2m-i}e_2^i, i = 0, \dots, 2m\}$. Moreover, for each $g \in \mathrm{SL}_2(\mathcal{O}_D)$ the matrix representing $\rho(m)(g)$ with respect to this basis has entries in $\mathcal{O}_D$. Thus if we let $L(m)$ be the $\mathbb{Z}$-module spanned by $\{e_1^{2m-i}e_2^i, i = 0, \dots, 2m\}$ and $\{\sqrt{-D}e_1^{2m-i}e_2^i, i = 0, \dots, 2m\}$ for $D \equiv 1, 2$ modulo 4 resp. $\{\frac{1+\sqrt{-D}}{2}e_1^{2m-i}e_2^i, i = 0, \dots, 2m\}$ for $D \equiv 3$ modulo 4 , then $L(m)$ is a lattice in $V_\mathbb{R}(m)$ which is preserved by $\rho_\mathbb{R}(m)(\Gamma(D))$. We shall denote the associated representation of $\Gamma(D)$ on $L(m)$ by $\rho_\mathbb{Z}(m)$. In order to make the basis considered by Menal-Ferrer and Porti into a basis consisting of integral elements, we need the following lemma.

Lemma 4.2. *Let $B_\eta \in \mathrm{SL}_2(F)$ and let $P_\eta := B_\eta^{-1} P B_\eta$, where $P = MAN$ is the standard parabolic subgroup of $\mathrm{SL}_2(\mathbb{C})$ as above. Let $N_\eta := B_\eta^{-1} N B_\eta$. Then there exist vectors*

$\omega(m)$, $\omega'(m) \in L(m)$, which are fixed by $\rho_{\mathbb{Z}}(m)\left(\pm(\Gamma(D) \cap N_\eta)\right)$ and which are linearly independent over $\mathbb{R}$.

Proof. The vectors $\omega(m) := e_1^{2m}$, $\omega'(m) := \sqrt{-D}e_1^{2m}$ for $D \equiv 1, 2$ modulo 4 resp. $\omega'(m) := \frac{1+\sqrt{-D}}{2}e_1^{2m}$ for $D \equiv 3$ modulo 4 belong to $L(m)$, are linearly independent over $\mathbb{R}$ and are fixed by $\rho_{\mathbb{Z}}(m)\left(\pm(\Gamma(D) \cap N)\right)$. The matrix representing $\rho_{\mathbb{R}}(m)(B_\eta^{-1})$ with respect to a base of $V_{\mathbb{R}}(m)$ coming from a $\mathbb{Z}$-base of $L(m)$ has entries in $\mathbb{Q}$. Thus multiplying $\rho_{\mathbb{R}}(m)(B_\eta^{-1})\omega(m)$, $\rho_{\mathbb{R}}(m)(B_\eta^{-1})\omega'(m)$ by the denominator of this matrix gives the desired vectors for N_η. $\square$

5. Proof of the main results

In this section we prove our main results. We start with the 0-th homology group. This group is pure torsion. Moreover, for sequences of congruence subgroups the asymptotic behaviour of its size was estimated directly by Raimbault.

Proposition 5.1. *Let Γ_i be a sequence of congruence subgroups of $\Gamma(D)$ such that $\lim_{i\to\infty}[\Gamma(D) : \Gamma_i] = \infty$. Then for every $m \in \mathbb{N}$ with $m \geq 1$ one has*

$$\lim_{i\to\infty} \frac{\log |H_0(\Gamma_i; L(m))|}{[\Gamma(D) : \Gamma_i)]} = 0.$$

Proof. This is proved by Raimbault, see [Ra1, Lemma 6.11]. $\square$

Now we have to estimate the volume factors with respect to the bases in the integral homology given by Menal-Ferrer and Porti which occur in Lemma 3.2. For the moment, we consider any X of the form $X = \Gamma\backslash\mathbb{H}^3$, where Γ is a discrete, torsion-free subgroup of $\Gamma(D)$ of finite index. Let $\partial\overline{X}$ be the boundary of $\overline{X}$ and let $\iota : \partial\overline{X} \to \overline{X}$ be the inclusion. Let $\mathrm{pr} : H_1(\overline{X}; L(m)) \to H_1(\overline{X}; L(m))_{free}$ be the projection. Recall that $H_2(\overline{X}; L(m))$ is free. It follows from Proposition 2.2 that $\iota_* H_2(\partial\overline{X}; L(m))$ resp. $\mathrm{pr}(\iota_* H_1(\partial\overline{X}; L(m))_{free})$ are lattices of finite index $[H_2(\overline{X}; L(m)) : \iota_* H_2(\partial\overline{X}; L(m))]$ resp. $[H_1(\overline{X}; L(m))_{free} : \mathrm{pr}(\iota_* H_1(\partial\overline{X}; L(m))_{free})]$ in $H_2(\overline{X}; L(m))$ resp. $H_1(\overline{X}; L(m))_{free}$. These indices can be estimated as follows.

Lemma 5.2. *One can estimate*

$$[H_2(\overline{X}; L(m)) : \iota_* H_2(\partial\overline{X}; L(m))] \leq |H_0(\overline{X}; L^*(m))_{tors}|.$$

Moreover, one can estimate

$$[H_1(\overline{X}; L(m))_{free} : \mathrm{pr}(\iota_* H_1(\partial\overline{X}; L(m))_{free})] \leq |H_1(\overline{X}; L^*(m))_{tors}|.$$

Proof. We prove the first estimate. The second estimate can be proved in the same way. Since the sequence

$$H_2(\partial\overline{X}; L(m)) \to H_2(\overline{X}; L(m)) \to H_2(\overline{X}, \partial\overline{X}; L(m))$$

is exact and since $\iota_* H_2(\partial\overline{X}; L(m))$ is a lattice of finite index in $H_2(\overline{X}; L(m))$ by Proposition 2.2, the quotient $H_2(\overline{X}; L(m))/\iota_* H_2(\partial\overline{X}; L(m))$ embeds into $H_2(\overline{X}, \partial\overline{X}; L(m))_{tors}$. By Poincaré duality, [Wa, page 223-224] one has

$$H_2(\overline{X}, \partial\overline{X}; L(m)) \cong H^1(\overline{X}; L(m)).$$

By Lemma 3.1 one has

$$H^1(\overline{X}; L(m))_{tors} \cong H_0(\overline{X}; L(m)^*)_{tors}$$

and the first estimate follows. □

Now we come to the covering situation. We let X_0 be a fixed hyperbolic manifold of the form $X_0 = \Gamma_0\backslash\mathbb{H}^3$, where Γ_0 is a torsion-free subgroup of $\Gamma(D)$ of finite index. We let $\kappa(X_0)$ be the number of cusps of X_0 and we let $T_{0,1}, \dots, T_{0,\kappa(X_0)}$ be the boundary components of $\overline{X}_0$. For each $k = 1, \dots, \kappa(X_0)$ we fix, according to Lemma 4.2, vectors $\omega_{0,k}^1(m)$, $\omega_{0,k}^2(m)$, in $L(m)$ which are linearly independent over $\mathbb{R}$ and fixed by $\rho_{\mathbb{R}}(m)(\pi_1(T_{0,k}))$. We let X be a finite covering of X_0. We let $\kappa(X)$ be the number of cusps of X and we let T_j, $j = 1, \dots, \kappa(X)$ be the boundary components of $\overline{X}$. Each T_j covers a single boundary component $T_{0,k(j)}$, $1 \le k(j) \le \kappa(X_0)$ of $\overline{X}_0$. We let $[T_{0,k(j)} : T_j]$ be the index of this covering. Moreover, we let $\omega_j^1(m) := \omega_{0,k(j)}^1(m)$, $\omega_j^2(m) := \omega_{0,k(j)}^2(m)$.

Lemma 5.3. *Let $m \ge 1$. There exists a constant $C > 0$ such that for each hyperbolic manifold X which is a finite covering of X_0 and each $j = 1, \dots, \kappa(X)$, there exist non-trivial cycles $\theta_j \in H_1(T_j; \mathbb{Z})$ such that the free $\mathbb{Z}$-submodule*

$$\mathcal{M}_{\overline{X}}(m) := \bigoplus_{j=1}^{\kappa(X)} \mathbb{Z}\,\mathrm{pr}\,\iota_*(\theta_j \otimes \omega_j^1(m)) \oplus \mathbb{Z}\,\mathrm{pr}\,\iota_*(\theta_j \otimes \omega_j^2(m))$$

of $H_1(\overline{X}; L(m))_{free}$ satisfies

$$[\mathrm{pr}\,\iota_*(H_1(\partial\overline{X}; L(m))) : \mathcal{M}_{\overline{X}}(m)] \le C^{\kappa(X)} \prod_{j=1}^{\kappa(X)} [T_{0,k(j)} : T_j],$$

where $\mathrm{pr} : H_1(\overline{X}; L(m) \to H_1(\overline{X}; L(m))_{free}$ *denotes the projection onto the free part in the homology.*

Proof. We consider each $H_1(T_{0,k}; \mathbb{Z})$ as a lattice in $H_1(T_{0,k}; \mathbb{R})$ and fix an inner product defined by a $\mathbb{Z}$-basis of $H_1(T_{0,k}; \mathbb{Z})$. The arguments of Menal-Ferrer and Porti [MePo2, page 17] easily imply that $\iota_*(H_1(T_{0,k}; V_{\mathbb{R}}(m)))$ is a 2-dimensional real vector space generated by $\iota_*(\theta \otimes \omega_{0,k}^1(m))$, $\iota_*(\theta \otimes \omega_{0,k}^2(m))$ for each non-zero $\theta \in H_1(T_{0,k}; \mathbb{Z})$. Let $v_1(k), v_2(k) \in H_1(T_{0,k}; L(m))_{free}$, such that $\mathrm{pr}\,\iota_*(H_1(T_{0,k}; L(m))_{free})$ is the free $\mathbb{Z}$-module generated by $\iota_*(v_1(k)), \iota_*(v_2(k))$. Thus for every $\theta \in H_1(T_{0,k}; \mathbb{Z})$ which is not zero there exists an integral matrix $A_k(\theta) = (a_{\mu,\nu}^k(\theta))$ such that in the free homology one has

$$\iota_*(\theta \otimes \omega_{0,k}^1(m)) = a_{1,1}^k(\theta)\iota_*(v_1(k)) + a_{2,1}^k(\theta)\iota_*(v_2(k)),$$
$$\iota_*(\theta \otimes \omega_{0,k}^2(m)) = a_{1,2}^k(\theta)\iota_*(v_1(k)) + a_{2,2}^k(\theta)\iota_*(v_2(k)).$$

Moreover, the matrix $A_k(\theta)$ is invertible over $\mathbb{R}$ for each non-zero θ. Since the matrix entries of $A_k(\theta)$ are linear functions of θ, there exists a constant $C > 0$ such that for all $k = 1, \dots, \kappa(X_0)$ and all $\theta \in H_1(T_{0,k}; \mathbb{Z})$ one has

$$|\det A_k(\theta)| \le C|\theta|^2. \tag{5.14}$$

Now fix a boundary component T_j of $\partial\overline{X}$. Let $\pi : X \to X_0$ denote the covering. Then π induces a covering $\pi : T_j \to T_{0,k(j)}$ which in turn induces a homomorphism $\pi_* : H_1(T_j; \mathbb{Z}) \to H_1(T_{0,k(j)}; \mathbb{Z})$. By Minkowski's lattice-point theorem there exists a non-trivial $\theta_j \in H_1(T_j; \mathbb{Z})$ such that

$$|\pi_*(\theta_j)| \le 2\sqrt{[T_{0,k(j)} : T_j]}. \tag{5.15}$$

Again there exist $w_1(j), w_2(j) \in H_1(T_j; L(m))_{free}$ such that $\mathrm{pr}\,\iota_*(w_1(j))$, $\mathrm{pr}\,\iota_*(w_2(j))$ form a $\mathbb{Z}$-basis of $\mathrm{pr}\,\iota_*(H_1(T_j; L(m))_{free})$. Thus in the free homology one has

$$\iota_*(\theta_j \otimes \omega_j^1(m)) = b_{1,1}(\theta_j)\iota_*(w_1(j)) + b_{2,1}(\theta_j)\iota_*(w_2(j))$$
$$\iota_*(\theta_j \otimes \omega_j^2(m)) = b_{1,2}(\theta_j)\iota_*(w_1(j)) + b_{2,2}(\theta_j)\iota_*(w_2(j)),$$

where the matrix $B(\theta_j) = (b_{\mu,\nu}(\theta_j))$ is integral. By the above arguments, it is invertible over $\mathbb{R}$.

Now we consider the map $\pi_* : H_1(\overline{X}, V_{\mathbb{R}}(m)) \to H_1(\overline{X}_0, V_{\mathbb{R}}(m))$ which restricts to a map $\pi_* : H_1(\overline{X}, L(m))_{free} \to H_1(\overline{X}_0, L(m))_{free}$. Then we have

$$\pi_*\iota_* w_1(j) = d_{1,1}(j)\iota_* v_1(k(j)) + d_{2,1}(j)\iota_* v_2(k(j)),$$
$$\pi_*\iota_* w_2(j) = d_{2,1}(j)\iota_* v_1(k(j)) + d_{2,2}(j)\iota_* v_2(k(j)),$$

where $D(j) = (d_{\mu,\nu}(j))$ is an integral 2×2 matrix. We have

$$A_{k(j)}(\pi_*(\theta_j)) = D(j) \cdot B(\theta_j). \tag{5.16}$$

Thus $D(j)$ is invertible over $\mathbb{R}$. Thus we have using (5.14), (5.15) and (5.16):

$$[\mathrm{pr}\,\iota_*(H_1(T_j : L(m))_{free}) : \mathbb{Z}\iota_*(\theta_j \otimes \omega_j^1(m)) + \mathbb{Z}\iota_*(\theta_j \otimes \omega_j^2(m))] = |\det(B(\theta_j))|$$
$$\le |\det(A_{k(j)}(\pi_*\theta_j))| \le 4C[T_{0,k(j)} : T_j]$$

and the Lemma follows. $\square$

Now to treat the image of $H_2(\partial\overline{X}; L(m))$ we will need the following Lemma.

Lemma 5.4. *If for each j one chooses a generator η_j of $H_2(T_j; \mathbb{Z})$, the free $\mathbb{Z}$-module*

$$\mathcal{N}_{\overline{X}}(m) := \bigoplus_{j=1}^{\kappa(X)} \mathbb{Z}\iota_*(\eta_j \otimes \omega_j^1(m)) \oplus \mathbb{Z}\iota_*(\eta_j \otimes \omega_j^2(m))$$

of $H_2(\overline{X}; L(m))$ satisfies

$$[\iota_*(H_2(\partial\overline{X}; L(m))) : \mathcal{N}_{\overline{X}}(m)] \le C^{\kappa(X)} \prod_{j=1}^{\kappa(X)} [T_{0,k(j)} : T_j].$$

Proof. The proof is entirely analogous to the proof of the preceding Lemma. □

Combining Lemma 5.2, Lemma 5.3 and Lemma 5.4, we can prove the following estimate for the volume factors.

Proposition 5.5. *Let Γ_0 be a congruence subgroup of $\Gamma(D)$ and assume that Γ_0 is torsion-free. Let Γ_i be a sequence of congruence subgroups of $\Gamma(D)$ contained in Γ_0 such that*

$$\lim_{i\to\infty}\frac{\kappa(\Gamma_i)\log[\Gamma_0:\Gamma_i]}{[\Gamma_0:\Gamma_i]}=0.$$

Let $X_i:=\Gamma_i\backslash\mathbb{H}^3$. Let $T_{i,j}$, $j=1,\dots,\kappa(\Gamma_i)$ denote the boundary components of X_i. Then for each $m\in\mathbb{N}$ with $m\geq 1$, each i and each j one can choose non-trivial cycles $\theta_{i,j}\in H_1(T_{i,j};\mathbb{Z})$ and vectors $\omega_{i,j}(m)$ fixed by $\rho_{\mathbb{Z}}(m)(T_{i,j})$ such that if for $\theta_i=\{\theta_{i,1},\dots,\theta_{i,\kappa(\Gamma_i)}\}$, $\omega_i(m)=\{\omega_{i,1}(m),\dots,\omega_{i,\kappa(\Gamma_i)}(m)\}$ the set $\mathcal{B}^{\mathbb{R}}(\theta_i,\omega_i(m))$ is the basis of $H_1(\overline{X}_i;\rho_{\mathbb{R}}(m))$ as in (2.8) and (3.11), one has

$$0\leq-\frac{\log\operatorname{vol}_{\mathcal{B}^{\mathbb{R}}(\theta_i,\omega_i(m))}H_1(\overline{X}_i;L(m))_{free}}{[\Gamma_0:\Gamma_i]}+a_i\leq\frac{\log|H_1(\Gamma_i;L^*(m))_{tors}|}{[\Gamma_0:\Gamma_i]},$$

where $a_i\in\mathbb{R}$ with $\lim_{i\to\infty}a_i=0$. Moreover, if for each i and each j one chooses a generator $\eta_{i,j}$ of $H_2(T_{i,j};\mathbb{Z})$, then for $\eta_i=\{\eta_{i,1},\dots,\eta_{i,\kappa(\Gamma_i)}\}$, and the basis $\mathcal{B}^{\mathbb{R}}(\eta_i,\omega_i(m))$ of $H_2(\overline{X}_i;\rho_{\mathbb{R}}(m))$ defined by (2.8) and (3.11), one has

$$\lim_{i\to\infty}\frac{\log\operatorname{vol}_{\mathcal{B}^{\mathbb{R}}(\eta_i,\omega_i(m))}H_2(\overline{X}_i;L(m))}{[\Gamma_0:\Gamma_i]}=0.$$

Proof. We choose the $\theta_{i,j}\in H_1(T_{i,j};\mathbb{Z})$ and the vectors $\omega^1_{i,j}(m),\omega^2_{i,j}\in L(m)$ as in Lemma 5.3. We let $\omega_{i,j}(m):=\omega^1_{i,j}(m)$. Let

$$\widetilde{\mathcal{B}}^{\mathbb{R}}(\theta_i,\omega^1_i(m),\omega^2_i(m)):=\bigsqcup_{j=1}^{\kappa(X_i)}\{\iota_*(\omega^1_{i,j}(m)\otimes\theta_{i,j}),\iota_*(\omega^2_{i,j}(m)\otimes\theta_{i,j})\}$$

Then $\widetilde{\mathcal{B}}^{\mathbb{R}}(\theta_i,\omega^1_i(m),\omega^2_i(m))$ is a basis of $H_1(X_i,V_{\mathbb{R}}(m))$ and if $M(i)$ is the matrix of base-change from $\widetilde{\mathcal{B}}^{\mathbb{R}}(\theta_i,\omega^1_i(m),\omega^2_i(m))$ to $\mathcal{B}^{\mathbb{R}}(\theta_i,\omega_i(m))$ one has $|\det M(i)|=\tilde{C}^{\kappa(\overline{X}_i)}$, where $\tilde{C}\in\mathbb{R}^+$ is a constant which is independent of X_i. Thus it suffices to estimate the term $-\log\operatorname{vol}_{\widetilde{\mathcal{B}}^{\mathbb{R}}(\theta_i,\omega^1_i(m),\omega^2_i(m))}H_1(\overline{X}_i;L(m))_{free}$. If $\mathcal{M}_{\overline{X}_i}(m)$ is as in Lemma 5.3 one has

$$\operatorname{vol}_{\widetilde{\mathcal{B}}^{\mathbb{R}}(\theta_i,\omega^1_i(m),\omega^2_i(m))}H_1(\overline{X}_i;L(m))_{free}=\frac{1}{[H_1(\overline{X}_i:L(m))_{free}:\mathcal{M}_{\overline{X}_i}(m)]}.$$

By Lemma 5.2 and Lemma 5.3 one can estimate

$$\begin{aligned}1\leq&[H_1(\overline{X}_i:L(m))_{free}:\mathcal{M}_{\overline{X}_i}(m)]\\=&[H_1(\overline{X}_i:L(m))_{free}:\operatorname{pr}(\iota_*(H_1(\partial\overline{X}_i;L(m))_{free}))][\operatorname{pr}(\iota_*(H_1(\partial\overline{X}_i;L(m))_{free})):\mathcal{M}_{\overline{X}_i}(m)]\\\leq&|H_1(\overline{X}_i;L^*(m))_{tors}|\cdot C^{\kappa(\Gamma_i)}\prod_{j=1}^{\kappa(\Gamma_i)}[T_{0,m(j)}:T_{i,j}].\end{aligned}$$

Thus we can estimate

$$\begin{aligned}0 &\leq -\log \operatorname{vol}_{\widetilde{\mathcal{B}}^{\mathbb{R}}(\theta_i,\omega_i^1(m),\omega_i^2(m))} H_1(\overline{X}_i; L(m))_{free} \\ &\leq \log |H_1(\overline{X}_i; L^*(m))_{tors}| + \kappa(\Gamma_i)\log C + \kappa(\Gamma_i)\log[\Gamma(D):\Gamma_i]\end{aligned}$$

and the first estimate follows. Applying Proposition 5.1 and Lemma 5.4, the second estimate can be proved in the same way. □

Now we can prove Theorem 1.1. We let the sequence $X_i = \Gamma_i\backslash\mathbb{H}^3$ be as in Theorem 1.1 resp. the previous proposition. We let $V_{\mathbb{R}}^*(m)$ resp. $V^*(m)$ be the dual spaces of $V_{\mathbb{R}}(m)$ resp. $V(m)$ and we let $\check{\rho}_{\mathbb{R}}(m) : \mathrm{SL}_2(\mathbb{C}) \to V_{\mathbb{R}}^*(m)$ resp. $\check{\rho}(m) : \mathrm{SL}_2(\mathbb{C}) \to V^*(m)$ be the contragredient representation of $\rho_{\mathbb{R}}(m)$ resp. $\rho(m)$. Then the representation $\check{\rho}(m)$, regarded as a real representation, is equivalent to $\check{\rho}_{\mathbb{R}}(m)$. Moreover, $\check{\rho}(m)$ is self-contragredient. Thus, if for $\mu \in \{2, m\}$ we fix vectors $\omega_i(\mu) \in L(\mu)$ as in the previous proposition and if we let $\omega_i(\mu)^* \in L(\mu)^*$ be the dual vectors, then we obtain bases $\mathcal{B}^{\mathbb{R}}(\theta_i, \omega_i^*(\mu))$ resp. $\mathcal{B}^{\mathbb{R}}(\eta_i, \omega_i^*(\mu))$ of $H_1(\overline{X}_i; V_{\mathbb{R}}^*(\mu))$ resp. of $H_2(\overline{X}_i; V_{\mathbb{R}}^*(\mu))$ for which the previous proposition continues to hold. Now we let $\bar{\rho}_{\mathbb{R}}(\mu) := \rho_{\mathbb{R}}(\mu) \oplus \check{\rho}_{\mathbb{R}}(\mu)$ acting on $\overline{V}_{\mathbb{R}}(\mu) := V_{\mathbb{R}}(\mu) \oplus V_{\mathbb{R}}^*(\mu)$. Then we obtain bases $\mathcal{B}^{\mathbb{R}}(\theta_i, \omega_i(\mu)) \sqcup \mathcal{B}^{\mathbb{R}}(\theta_i, \omega_i^*(\mu))$, $\mathcal{B}^{\mathbb{R}}(\eta_i, \omega_i(\mu)) \sqcup \mathcal{B}^{\mathbb{R}}(\eta_i, \omega_i^*(\mu))$ of $H_1(\overline{X}_i, \overline{V}_{\mathbb{R}}(\mu))$ resp. of $H_2(\overline{X}_i, \overline{V}_{\mathbb{R}}(\mu))$. To save notation, we write $\mathcal{B}_1(\mu) := \mathcal{B}^{\mathbb{R}}(\theta_i, \omega_i(\mu)) \sqcup \mathcal{B}^{\mathbb{R}}(\theta_i, \omega_i^*(\mu))$, $\mathcal{B}_2(\mu) := \mathcal{B}^{\mathbb{R}}(\eta_i, \omega_i(\mu)) \sqcup \mathcal{B}^{\mathbb{R}}(\eta_i, \omega_i^*(\mu))$.
We identify $H_*(\overline{X}_i; L(m)) \cong H_*(\Gamma_i; L(m))$. By Proposition 5.5 one has

$$\begin{aligned}&2\liminf_{i\to\infty} \frac{\log|H_1(\Gamma_i; \overline{L}(m))_{tors}|}{\operatorname{vol}(X_i)} \\ &\geq \liminf_{i\to\infty} \frac{\log|H_1(\Gamma_i; \overline{L}(m))_{tors}| - \log \operatorname{vol}_{\mathcal{B}_1(m)} H_1(\Gamma_i; \overline{L}(m))_{free}}{\operatorname{vol}(X_i)}.\end{aligned}$$

On the other hand, by Proposition 5.5 and Proposition 5.1 one has

$$\begin{aligned}&\liminf_{i\to\infty} \frac{\log|H_1(\Gamma_i; \overline{L}(m))_{tors}| - \log \operatorname{vol}_{\mathcal{B}_1(m)} H_1(\Gamma_i; \overline{L}(m))_{free}}{\operatorname{vol}(X_i)} \\ &\geq \liminf_{i\to\infty}\Bigg(\frac{\log|H_1(\Gamma_i; \overline{L}(m))_{tors}| - \log \operatorname{vol}_{\mathcal{B}_1(m)} H_1(\Gamma_i; \overline{L}(m))_{free} - \log|H_0(\Gamma_i; \overline{L}(m))_{tors}|}{\operatorname{vol}(X_i)} \\ &+ \frac{\log \operatorname{vol}_{\mathcal{B}_2(m)}(H_2(\Gamma_i; \overline{L}(m))_{free}) - \log|H_1(\Gamma_i; \overline{L}(2))_{tors}| + \log \operatorname{vol}_{\mathcal{B}_1(2)} H_1(\Gamma_i; \overline{L}(2))_{free}}{\operatorname{vol}(X_i)} \\ &+ \frac{\log|H_0(\Gamma_i; \overline{L}(2))_{tors}| - \log \operatorname{vol}_{\mathcal{B}_2(2)}(H_2(\Gamma_i; \overline{L}(2))_{free})}{\operatorname{vol}(X_i)}\Bigg).\end{aligned}$$

By Lemma 3.2, Lemma 3.3 and since $\rho(m)$ is self-contragredient, for each $m \in \mathbb{N}$, the last $\liminf$ equals

$$\liminf_{i\to\infty} -4\frac{\log \mathcal{T}_{\overline{X}_i}(\rho(m))}{\operatorname{vol}(X_i)}.$$

If $\lim_{i\to\infty}[\Gamma(D):\Gamma_i]=\infty$, then $\lim_{i\to\infty}\ell(\Gamma_i)=\infty$ and thus by Proposition 2.3 one has

$$\liminf_{i\to\infty} -\log \mathcal{T}_{\overline{X}_i}(\rho(m)) = \frac{m(m+1)-6}{\pi}.$$

This proves Theorem 1.1. □

To prove Corollary 1.2, we remark that by (4.12) and Lemma 4.1 we can estimate

$$\frac{\kappa(\Gamma(\mathfrak{a}_i))\log[\Gamma(D):\Gamma(\mathfrak{a}_i)]}{[\Gamma(D):\Gamma(\mathfrak{a}_i)]} \leq \frac{3d_F\log N(\mathfrak{a}_i)}{N(\mathfrak{a}_i)}.$$

and thus Corollary 1.2 follows from Theorem 1.1.

Remark 5.6. At the end of this article we want to remark that in our oppinion the assumption made in Theorem 1.1 that all groups Γ_i are contained in a torsion-free subgroup Γ_0 of $\Gamma(D)$ is probably unneccessary. We think that Theorem 1.1 and its proof presented here can be generalized to any sequence of congruence subgroups Γ_i of $\Gamma(D)$ satisfying (1.3). The assumption that all Γ_i are contained in Γ_0 was only used in the proof of Lemma 5.3 and Lemma 5.4 where we used the construction of an explicit basis in the homology with twisted coefficients for the manifold X_0 given by Menal-Ferrer and Porti in [MePo2, Proposition 2.2]. We think that these results of Menal-Ferrer and Porti can in turn be generalized to the space $\Gamma(D)\backslash\mathbb{H}^3$ which is not a manifold but only a good hyperbolic orbifold of finite volume.

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